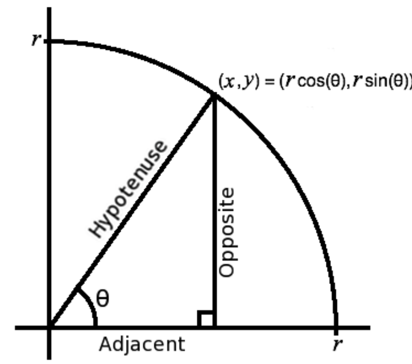
Trigonometry II

Campus Academic Resource Program

Remembering the Definitions

A useful acronym to remember the trig definitions is **SOH-CAH-TOA**. SOH: **S**ine is **O**pposite over **H**ypotenuse. CAH: **C**osine is **A**djacent over **H**ypotenuse. TOA: **T**angent is **A**djacent over **O**pposite.



Deriving the Pythagorean Identities

This Follows directly from the Pythagorean Theorem $x^2 + y^2 = r^2$. In this case notice $x = r\cos(\theta)$ and $y = r\sin(\theta)$. Try plugging these into the Pythagorean Theorem to get the 1st Pythagorean Identity.

Try to derive the 2nd Pythagorean Identity by starting with the 1st Identity and dividing both sides by $\cos^2(\theta)$.

Try to derive the 3rd Pythagorean Identity by starting with the 1st Identity and dividing both sides by $\sin^2(\theta)$.

Deriving the Sum-Angle Identity

Recall from algebra "Re" is taking the real part of a complex number, which is just ignoring the stuff with the *i*. And "Im" is taking the imaginary part of a complex number, which is just ignoring the stuff without the *i*. In other words, $\mathrm{Re}(x+iy) = x$ and $\mathrm{Im}(x+iy) = y$.

The Sum-angle Identities can be derived from Euler's Formula, $e^{i\theta} = \cos(\theta) + i\sin(\theta)$. First notice $\mathrm{Re}(e^{i\theta}) = \mathrm{Re}[\cos(\theta) + i\sin(\theta)] = \cos(\theta)$. So to find $\cos(\theta + \phi)$ you need to find $\mathrm{Re}[e^{i(\theta + \phi)}]$

$$\cos(\theta+\varphi) = \mathrm{Re}\left[e^{i(\theta+\varphi)}\right]$$

$= \mathrm{Re}\left[e^{i\theta}e^{i\varphi}\right]$ Now use Euler's Formula on $e^{i\theta}$.and $e^{i\phi}$

$= \mathrm{Re}[\cos\theta + i\sin\theta\cos\varphi + i\sin\varphi]$ Now FOIL carefully

$= \mathrm{Re}[\cos\theta\cos\varphi + i\cos\theta\sin\varphi + i\cos\varphi\sin\theta + i^2\sin\theta\sin\varphi]$Recall $i^2 = -1$

$$= \mathrm{Re}[\cos(\theta)\cos(\varphi) - \sin(\theta)\sin(\varphi)] + i[\cos(\theta)\sin(\varphi) + \cos(\varphi)\sin(\theta)]$$

$$= \cos(\theta)\cos(\varphi) - \sin(\theta)\sin(\varphi)$$

Try deriving the Sum-Angle Identity starting with $\sin(\theta + \phi) = \mathrm{Im}[e^{i(\theta + \phi)}]$. Now with the Sum-Angle Identities for sine & cosine we can derive the identity for tangent. Start with:

$$\tan(\theta+\varphi) = \frac{\sin(\theta+\varphi)}{\cos(\theta+\varphi)} = \frac{\sin(\theta)\cos(\varphi)+\sin(\varphi)\cos(\theta)}{\cos(\theta)\cos(\varphi)-\sin(\theta)\sin(\varphi)} = \frac{\frac{\sin(\theta)\cos(\varphi)+\sin(\varphi)\cos(\theta)}{1}}{\frac{\cos(\theta)\cos(\varphi)-\sin(\theta)\sin(\varphi)}{1}}$$

$$= \frac{\frac{\sin(\theta)\cos(\varphi)+\sin(\varphi)\cos(\theta)}{1}}{\frac{\cos(\theta)\cos(\varphi)-\sin(\theta)\sin(\varphi)}{1}} \cdot \frac{\frac{1}{\cos(\theta)\cos(\varphi)}}{\frac{1}{\cos(\theta)\cos(\varphi)}}$$

The rest should be straight forward.

Deriving Double-Angle Identities

To derive these simply use the Sum-angle identity and set ϕ equal to θ so that you have $\sin(\theta + \theta)$, $\cos(\theta + \theta)$ and $\tan(\theta + \theta)$. Try to derive all 5 identities. Notice there are three identities for cosine. To derive the 2nd and 3rd $\cos(2\theta)$ identity. Recall the Pythagorean Identity, $\cos^2(\theta) + \sin^2(\theta) = 1$, from this notice $\cos^2(\theta) = 1 - \sin^2(\theta)$ and $\sin^2(\theta) = 1 - \cos^2(\theta)$.

Deriving Half-Angle Identities

Try replacing θ with $\phi/2$ in the Double-Angle Identities for cosine and solving for $\cos(\phi/2)$ or $\sin(\phi/2)$. To get the identity for tangent remember tangent is sine over cosine. Try deriving those 3 identities.

Deriving Special Angles

To derive the value for sin(30°) = sin(π/6) and cos(30°) = cos(π/6) consider an equilateral triangle cut down the middle like in Fig. 1.

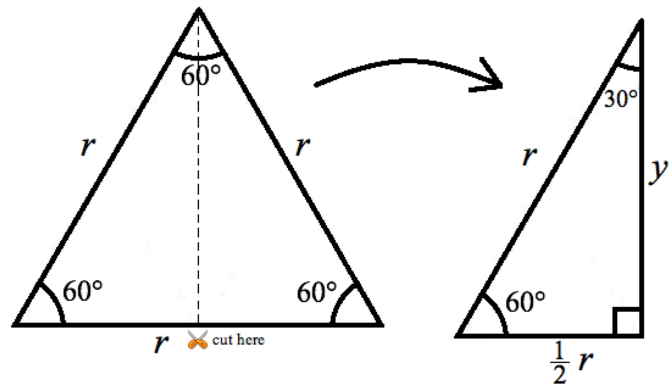


Fig. 1: *60° and 30° Triangle*

$$\sin(30°) = \sin\frac{\pi}{6} = \frac{Opp}{Hyp} = \frac{\frac{1}{2}r}{r} = \frac{1}{2}$$

For cos(30°) = cos(π/6) is is slightly more difficult since we need to know the *y* value in the picture but we can figure that out with Pythagoreans Theorem.

$$\frac{1}{2}r^{2} + y^2 = r^2 \text{so} y = \sqrt{r^2 - \frac{1}{4}r^2} = \sqrt{\frac{3}{4}r^2} = \frac{\sqrt{3}}{2}r$$

No that we know *y* we can find cos(30°) = cos(π/6)

$$\cos(30°) = \cos(\pi/6) = \frac{Adj}{Hyp} = \frac{\frac{\sqrt{3}}{2}r}{r} = \frac{\sqrt{3}}{2}$$

Try deriving sin(60°), cos(60°), tan(60°) and tan(30°) using Fig. 1 and try to derive sin(45°), cos(45°) and tan(45°) using Fig.2.

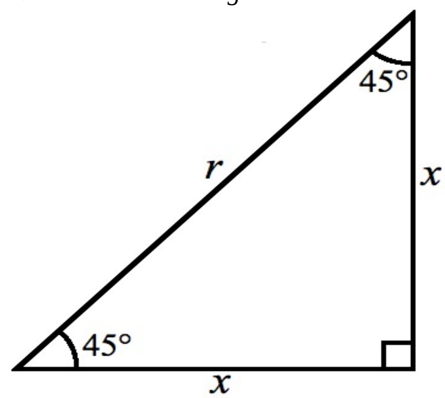


Fig. 2: Right *Isosceles Triangle*

Law of Sines

Consider a general triangle ABC in Fig 3. Notice that:

$$\sin(C) = \frac{h}{b} \text{and} \sin(B) = \frac{h}{c}$$

Solving for *h* gives:

$$h = b \cdot \sin(C) \text{and} h = c \cdot \sin(B) \text{so}$$

$$b \cdot \sin(C) = c \cdot \sin(B) \text{or} \frac{\sin(B)}{b} = \frac{\sin(C)}{c}$$

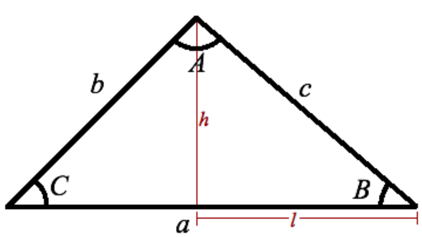


Fig. 3: *Triangle ABC*

Law of Cosine

Start with the Pythagorean Theorem. Notice: $c^2 = h^2 + l^2$We can put h and l in terms of *a*, *b* and *C* since: $h = b \cdot \sin(C)$and$l = a - b \cdot \cos(C)$Therefore:

$$c^2 = h^2 + l^2 = b \cdot \sin(C)^2 + a - b \cdot \cos(C)^2$$
$$= b^2\sin^2(C) + a^2 - 2a \cdot b \cdot \cos(C) + b^2\cos^2(C)$$
$$= a^2 + b^2\sin^2(C) + \cos^2(C) - 2a \cdot b \cdot \cos(C)$$
$$= a^2 + b^2 - 2a \cdot b \cdot \cos(C)$$

Area of a Triangle

Recall for a triangle$Area = \frac{1}{2}baseheight$. In the case of Fig. 3:

$$Area = \frac{1}{2}a \cdot h = \frac{1}{2}a \cdot b \cdot \sin(C)$$

If you have any question about the material feel free to come to CARP for free one on one tutoring.