Infinite Series

CARP
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Test for Divergence

If $\lim_{n\to\infty} a_n \neq 0$ or $\lim_{n\to\infty} a_n$ does not exist then $\sum_{n=1}^{\infty} a_n$ does not converge.

> **Note**: *This test only shows divergence, so you can't use it to show something converges*. For example, if $a_n = \frac{1}{n}$, then $\frac{1}{n} \to 0$, but $\sum \frac{1}{n}$ diverges!

Geometric Series

A *geometric series* is a series of the form $\sum_{n=0}^{\infty} a r^n$. It converges if $|r|<1$, and diverges otherwise. Furthermore, if $|r|<1$, then:

$$\sum_{n=0}^{\infty} a r^n = \frac{a}{1-r}$$

Integral Test

Suppose f is continuous, positive and decreasing on $[1,\infty)$. If $f(n) = a_n$, then $\sum_{n=1}^{\infty} a_n$ converges if and only if $\int_1^{\infty} f(n)\,dn$ converges.

p-Series

The p-series, $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if $p>1$ and diverges otherwise.

Comparison Test

If $\sum a_n$ and $\sum b_n$ are series with positive terms then:

(i) If $a_n \le b_n$ for large n and $\sum b_n$ converges, then $\sum a_n$ also converges.

(ii) If $a_n \le b_n$ for large n and $\sum a_n$ diverges, then $\sum b_n$ also diverges.

Limit Comparison Test

If $\sum a_n$ and $\sum b_n$ are series with positive terms.

(i) If $0 < \lim_{n\to\infty} \frac{a_n}{b_n} < \infty$, then $\sum a_n$ converges if and only if $\sum b_n$ converges.

(ii) If $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$ and $\sum b_n$ converges, then $\sum a_n$ also converges.

(iii) If $\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$ and $\sum b_n$ diverges, then $\sum a_n$ also diverges.

Alternating Series Test

An alternating series of the form $\sum (-1)^n b_n$ converges if b_n is a sequence that decreases to zero for large n.

Absolute Convergence

An absolutely convergent series is also convergent.

Ratio Test

Let $L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$. If $L < 1$, then $\sum a_n$ converges absolutely. If $L > 1$, then $\sum a_n$ diverges. If $L = 1$, then the test is inconclusive.

Root Test

Let $L = \lim_{n\to\infty} \sqrt[n]{a_n}$. If $L < 1$, then $\sum a_n$ converges absolutely. If L > 1, then $\sum a_n$ diverges. If L = 1, then the test is inconclusive.

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Properties of Series: Let a_n, b_n be real sequences and let c be a real number.

1. $\sum_{k=0}^{\infty} a_n + \sum_{k=0}^{\infty} b_n = \sum_{k=0}^{\infty} a_n + b_n$

2. $\sum_{k=0}^{\infty} a_n - \sum_{k=0}^{\infty} b_n = \sum_{k=0}^{\infty} a_n - b_n$

3. $\sum_{k=0}^{\infty} c \cdot a_n = c \cdot \sum_{k=0}^{\infty} a_n$

4. $\sum_{n=0}^{\infty} a_n \cdot \sum_{n=0}^{\infty} b_n = \sum_{n=0}^{\infty} \sum_{k=0}^{n} a_k \cdot b_{n-k}$

Taylor Series

Let $f(x)$ be infinitely differentiable on an interval I containing a real number a, then $f(x)$ has a power series expansion. Moreover,

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

Common Taylor Expansions

1. $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$
2. $\ln(x) = \sum_{n=0}^{\infty} \frac{(-1)^{n-1}}{n}(x-1)^n$
3. $\cos(x) = \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$
4. $\sin(x) = \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$
5. $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$
6. $\tan^{-1}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1}$

Tips for Testing Series.

1. *The test for divergence should be the first test you use!* If you don't you could waste a lot time trying a bunch of tests that wont work before you realize it doesn't pass the test for divergence.

2. Seeing a $(-1)^n$ in the summation should tip you off to use alternating series test, but be aware that sin($2\pi n$) and cos($2\pi n$) also make an alternating series.

3. If you see everything in the summand has an exponent of n then try the root test.

4. If there are factorials in the summand or exponents of n on some of the terms, use the ratio test.

5. If you have a rational polynomial try the comparison test and compare it to the largest term of the numerator over the largest term of the denominator.

6. A summation with a natural log can be tested using the limit comparison test and comparing it to the same function, but with $\sqrt{n}$ replacing ln(n). It can also be tested with the integral test if it is easily integrable.

7. If sin(x) and cos(x) appear you can use the comparison test and the fact that $-1 \le \cos(x) \le 1$ or $-1 \le \sin(x) \le 1$.

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