Series Quiz

Campus Academic Resource Program

1. Derive the taylor expansion for $f(x) = \cos(x)$.

2. Test the Series for convergence. If possible find the sum.

   a) $\sum_{n=1}^{\infty} \frac{e^n}{3^{n-1}}$

   b) $\sum_{n=1}^{\infty} n^{-3}\cos^2(n)$

   c) $\sum_{n=0}^{\infty} \frac{\arctan(n)}{1+n^2}$

   d) $\sum_{n=0}^{\infty} \frac{(2n)!}{3^n}$

   e) $\sum_{n=1}^{\infty} \frac{\ln(n)}{n^5}$

   f) $\sum_{n=0}^{\infty} \left(1 - \frac{1}{n}\right)^n$

3. Find the taylor expansion for $f(x) = x^2 \sin\left(\frac{\pi}{x}\right)$.

Series Quiz

Campus Academic Resource Program

Quiz Solutions

1. Derive the taylor expansion for $f(x) = \cos(x)$.

$$
\begin{aligned}
f(x) &= \cos(x) & f(0) &= 1 \\
f'(x) &= -\sin(x) & f'(0) &= 0 \\
f''(x) &= -\cos(x) & f''(0) &= -1 \\
f^{(3)}(x) &= \sin(x) & f^{(3)}(0) &= 0 \\
f^{(4)}(x) &= \cos(x) & f^{(4)}(0) &= 1 \\
&\vdots & &\vdots
\end{aligned}
$$

So notice all the even derivatives alternate and the odd terms are zero. So we only want the even terms in the series since all the odd terms are multiplied by 0. We do this by replacing n with $2n$. Then we add $(-1)^n$ to make it alternate properly, giving:

$$\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$$

2. Test the Series for convergence. If possible find the sum.

   a) $\displaystyle\sum_{n=1}^{\infty} \frac{e^n}{3^{n-1}}$

$$
\begin{aligned}
\sum_{n=1}^{\infty} \frac{e^n}{3^{n-1}} &= \sum_{n=1}^{\infty} \frac{e^n}{3^n 3^{-1}} \\
&= \sum_{n=1}^{\infty} 3 \frac{e^n}{3^n} \\
&= 3 \sum_{n=1}^{\infty} \left(\frac{e}{3}\right)^n
\end{aligned}
$$

This is a geometric series! $\frac{e}{3} < 1$ So the series converges so we can find the sum also:

$$
\begin{aligned}
&= 3 \sum_{n=1}^{\infty} \left(\frac{e}{3}\right)^n \\
&= \frac{3}{1-e/3} \\
&= \frac{9}{3-e}
\end{aligned}
$$

Series Quiz

Campus Academic Resource Program

b) $\sum_{n=1}^{\infty} n^{-3}\cos^2(n)$

$$\sum_{n=1}^{\infty} n^{-3}\cos^2(n) = \sum_{n=1}^{\infty} \frac{\cos^2(n)}{n^3}$$

We can use the comparison test and the fact that $\cos^2(n) \le 1$. So:

$$\sum_{n=1}^{\infty} \frac{\cos^2(n)}{n^3} \le \sum_{n=1}^{\infty} \frac{1}{n^3}$$

As you can see, $\sum_{n=1}^{\infty} \frac{1}{n^3}$ is a *p*-series with $p = 3$ so it converges. Therefore, $\sum_{n=1}^{\infty} \frac{\cos^2(n)}{n^3}$ converges as well.

c) $\sum_{n=0}^{\infty} \frac{\arctan(n)}{1+n^2}$

Notice that the derivative of arctan(*n*) appears in the summand so this is actually an easy integral using *u*-substitution.

$$\int_0^{\infty} \frac{\arctan(n)}{1+n^2}\,dn \qquad \text{Let } u = \arctan(n) \quad \text{and} \quad dn = (1+n^2)\,du$$

$$= \int_0^{\pi/2} \frac{u}{1+n^2}(1+n^2)\,du$$

$$= \int_0^{\pi/2} u\,du = \left[\frac{u^2}{2}\right]_{u=0}^{u=\frac{\pi}{2}}$$

$$= \frac{\pi^2}{8}$$

Since the integral converges the sum must also converge.

d) $\sum_{n=0}^{\infty} \frac{(2n)!}{3^n}$

The factorial in the problem should tell you to use the ratio test.

$$\lim_{n\to\infty}\left|\frac{(2(n+1))!}{3^{n+1}}\cdot\frac{3^n}{(2n)!}\right| = \lim_{n\to\infty}\left|\frac{(2n+2)!}{3^n 3^1}\cdot\frac{3^n}{(2n)!}\right|$$

$$= \lim_{n\to\infty}\left|\frac{(2n+2)(2n+1)}{3}\right|$$

$$= \infty$$

So the ratio test tells us the series diverges.

Series Quiz

Campus Academic Resource Program

e) $\sum_{n=1}^{\infty} \frac{\ln(n)}{n^5}$

With the ln(n) in the series, you can either try the integral test or the limit comparison test. The limit comparison test might be best in this case. We will compare this series to the series $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n^5}$ (Just replace ln(n) with $\sqrt{n}$).

$$\lim_{n\to\infty} \frac{\frac{\ln(n)}{n^5}}{\frac{\sqrt{n}}{n^5}} = \lim_{n\to\infty} \frac{\ln(n)}{\sqrt{n}} = \frac{\infty}{\infty} \quad \text{So we use L'Hospital's rule.}$$

$$= \lim_{n\to\infty} \frac{\ln(n)}{\sqrt{n}} = \lim_{n\to\infty} \frac{1/n}{\frac{1}{2}n^{-1/2}} = \lim_{n\to\infty} \frac{2\sqrt{n}}{n} = \lim_{n\to\infty} \frac{2}{\sqrt{n}} = 0$$

Therefore, by the limit comparison test the series converges.

f) $\sum_{n=0}^{\infty} \left(1 - \frac{1}{n}\right)^n$

If you tried the divergence test on this let $L = \lim_{n\to\infty} \left(1 - \frac{1}{n}\right)^n$

$$L = \lim_{n\to\infty} \left(1 - \frac{1}{n}\right)^n \quad \text{To calculate this limit log both sides}$$

$$\ln(L) = \ln\left(\lim_{n\to\infty} \left(1 - \frac{1}{n}\right)^n\right)$$

$$\ln(L) = \lim_{n\to\infty} n \ln\left(1 - \frac{1}{n}\right) \quad \text{This is a } \infty \cdot 0 \text{ case so we want to use}$$

L'Hospital's rule

$$\ln(L) = \lim_{n\to\infty} \frac{\ln(1 - 1/n)}{1/n} = \frac{0}{0} \quad \text{So applying L'Hospital's rule gives:}$$

$$\ln(L) = \lim_{n\to\infty} \frac{\frac{1}{1-1/n}\left(n^{-2}\right)}{\left(n^{-2}\right)}$$

$$\ln(L) = \lim_{n\to\infty} \frac{1}{1 - 1/n}$$

$$\ln(L) = 1 \quad \text{So,} \quad L = \lim_{n\to\infty} \left(1 - \frac{1}{n}\right)^n = e \neq 0$$

Therefore, it diverges by the divergence test.

Series Quiz

Campus Academic Resource Program

3. Find the taylor expansion for $f(x) = x^2 \sin\left(\frac{\pi}{x}\right)$.

Recall that $\sin(x) = \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$ so replacing x with $\frac{\pi}{x}$ you get:

$\sin\left(\frac{\pi}{x}\right) = \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n+1)!} \left(\frac{\pi}{x}\right)^{2n+1}$ now multiplying both sides by x^2

$$x^2 \sin\left(\frac{\pi}{x}\right) = \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n+1)!} \left(\frac{\pi}{x}\right)^{2n+1} x^2$$

$$x^2 \sin\left(\frac{\pi}{x}\right) = \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n+1)!} \left(\frac{\pi^{2n+1}}{x^{2n-1}}\right)$$