

Formulas and Theorems

1a. Definition of Limit: Let f be a function defined on an open interval containing c (except possibly at c) and let L be a real number. Then $\lim_{x \rightarrow a} f(x) = L$ means that for each $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$|f(x) - L| < \varepsilon \text{ whenever } 0 < |x - c| < \delta.$$

1b. A function $y = f(x)$ is continuous at $x = a$ if

- i). $f(a)$ exists
- ii). $\lim_{x \rightarrow a} f(x)$ exists
- iii). $\lim_{x \rightarrow a} = f(a)$

2. Even and Odd Functions

1. A function $y = f(x)$ is even if $f(-x) = f(x)$ for every x in the function's domain. Every even function is symmetric about the y-axis.
2. A function $y = f(x)$ is odd if $f(-x) = -f(x)$ for every x in the function's domain. Every odd function is symmetric about the origin.

3. Periodicity

A function $f(x)$ is periodic with period p ($p > 0$) if $f(x + p) = f(x)$ for every value of x .

Note: The period of the function $y = A \sin(Bx + C)$ or $y = A \cos(Bx + C)$ is $\frac{2\pi}{|B|}$.

The amplitude is $|A|$. The period of $y = \tan x$ is π .

4. Intermediate-Value Theorem

A function $y = f(x)$ that is continuous on a closed interval $[a, b]$ takes on every value between $f(a)$ and $f(b)$.

Note: If f is continuous on $[a, b]$ and $f(a)$ and $f(b)$ differ in sign, then the equation $f(x) = 0$ has at least one solution in the open interval (a, b) .

5. Limits of Rational Functions as $x \rightarrow \pm\infty$

- i). $\lim_{x \rightarrow \pm\infty} \frac{f(x)}{g(x)} = 0$ if the degree of $f(x) <$ the degree of $g(x)$

$$\text{Example: } \lim_{x \rightarrow \infty} \frac{x^2 - 2x}{x^3 + 3} = 0$$

- ii). $\lim_{x \rightarrow \pm\infty} \frac{f(x)}{g(x)}$ is infinite if the degrees of $f(x) >$ the degree of $g(x)$

$$\text{Example: } \lim_{x \rightarrow \infty} \frac{x^3 + 2x}{x^2 - 8} = \infty$$

- iii). $\lim_{x \rightarrow \pm\infty} \frac{f(x)}{g(x)}$ is finite if the degree of $f(x) =$ the degree of $g(x)$

$$\text{Example: } \lim_{x \rightarrow \infty} \frac{2x^2 - 3x + 2}{10x - 5x^2} = -\frac{2}{5}$$

6. Horizontal and Vertical Asymptotes

1. A line $y = b$ is a horizontal asymptote of the graph $y = f(x)$ if either

$$\lim_{x \rightarrow \infty} f(x) = b \text{ or } \lim_{x \rightarrow -\infty} f(x) = b.$$
2. A line $x = a$ is a vertical asymptote of the graph $y = f(x)$ if either

$$\lim_{x \rightarrow a^+} f(x) = \pm\infty \text{ or } \lim_{x \rightarrow a^-} f(x) = \pm\infty.$$

7. Average and Instantaneous Rate of Change

- i). Average Rate of Change: If (x_0, y_0) and (x_1, y_1) are points on the graph of $y = f(x)$, then the average rate of change of y with respect to x over the interval $[x_0, x_1]$ is

$$\frac{f(x_1) - f(x_0)}{x_1 - x_0} = \frac{y_1 - y_0}{x_1 - x_0} = \frac{\Delta y}{\Delta x}.$$

- ii). Instantaneous Rate of Change: If (x_0, y_0) is a point on the graph of $y = f(x)$, then the instantaneous rate of change of y with respect to x at x_0 is $f'(x_0)$.

8.
$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

9. The Number e as a limit

- i).
$$\lim_{n \rightarrow +\infty} \left(1 + \frac{1}{n}\right)^n = e$$

- ii).
$$\lim_{n \rightarrow 0} \left(1 + \frac{n}{1}\right)^{\frac{1}{n}} = e$$

10. Rolle's Theorem

If f is continuous on $[a, b]$ and differentiable on (a, b) such that $f(a) = f(b)$, then there is at least one number c in the open interval (a, b) such that $f'(c) = 0$.

11. Mean Value Theorem

If f is continuous on $[a, b]$ and differentiable on (a, b) , then there is at least one number c in (a, b) such that $\frac{f(b) - f(a)}{b - a} = f'(c)$.

12. Extreme-Value Theorem

If f is continuous on a closed interval $[a, b]$, then $f(x)$ has both a maximum and minimum on $[a, b]$.

13. To find the maximum and minimum values of a function $y = f(x)$, locate

1. the points where $f'(x)$ is zero or where $f'(x)$ fails to exist.
2. the end points, if any, on the domain of $f(x)$.

Note: These are the only candidates for the value of x where $f(x)$ may have a maximum or a minimum.

14. Let f be differentiable for $a < x < b$ and continuous for $a \leq x \leq b$,

1. If $f'(x) > 0$ for every x in (a, b) , then f is increasing on $[a, b]$.
2. If $f'(x) < 0$ for every x in (a, b) , then f is decreasing on $[a, b]$.